

Last Time: Inclusion-Exclusion
Pigeonhole Principle

Ex: Let $S = \{n \in \mathbb{Z} : 0 \leq n < 1000 \text{ and } n \text{ is even}\}$.

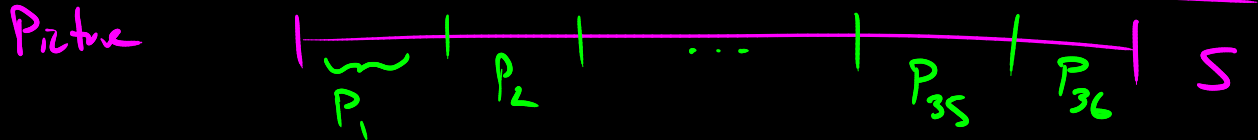
Every 38 elements of S contains a pair $a, b \in S$ distinct with $|a-b| \leq 26$.

Sol: Let $P_k := \{x \in S : 2 + (k-1)28 \leq x \leq 28k\}$

for $1 \leq k \leq 35$ and

$$P_{36} = \{x \in S : 982 \leq x\}$$

Picture



$\mathcal{P} = \{P_k : 1 \leq k \leq 36\}$ is a weak partition of

S with 36 parts. On the other hand,

for all $x, y \in P_k$ (for k fixed), we have

$$2 + (k-1)28 \leq x, y \leq 28k$$

Here $|x-y| \leq 28k - (28(k-1) + 2) = 26$.

Let $T \subseteq S$ have size 38. Because we have

36 parts in $\mathcal{P}$, some two elements of T belong

to the same part of $\mathcal{P}$, by the Pigeonhole Principle.

Hence, those two elements of T satisfy $|a-b| \leq 26$
as desired. $\square$

Relations and Functions

IDEA: Model relationships between elements of various sets...

Defⁿ: Let A, B be sets. A relation from A to B is any subset $R \subseteq A \times B$. We often write

* $[A \xrightarrow{R} B]$ to mean "R is a relation from A to B".
we also write " $a R b$ " to denote $(a, b) \in R$.

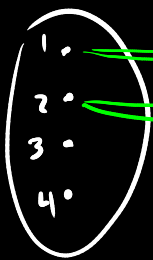
Ex: $R = \{(\underline{1}, 4), (2, \underline{4}), (\underline{1}, 3), (2, 5)\} \subseteq \underline{[4]} \times [7]$,

So $[4] \xrightarrow{R} [7]$. $1 R 4$, $1 R 3$.

NB: To represent relations via pictures:

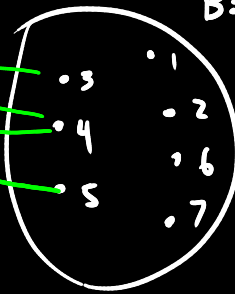
write

$A = [4]$



(arrows connect related elements...)

$B = [7]$



NB: A relation on set S is a relation $\underline{S} \xrightarrow{R} \underline{S}$.

Ex: Let $P = \{x : x \text{ is a person}\}$. The following are all relations on P :

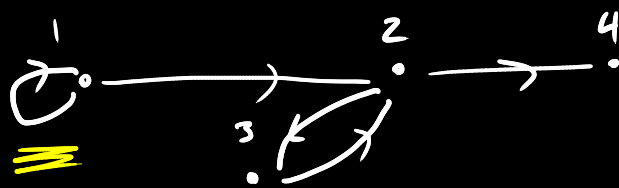
① $P \xrightarrow{\text{sis}} P$ where $(x, y) \in \text{sis}$ when x is a sister of y .

② $P \xrightarrow{\text{fri}} P$ where $x \text{ fri } y$ when x and y are mutual friends.

③ $P \xrightarrow{\text{Par}} P$ where $x \text{ Par } y$ when x is a parent of y .

NB: When we study relations on a set, we can simplify our diagrams, like so: just lay out the elements of S and draw arrows between them.

Ex: $R = \{(1,1), (1,2), (2,3), (3,2), (2,4)\} \subseteq [4] \times [4]$.
is a relation on $[4]$. The picture is:



Defn: A relation R on S is reflexive when for all $x \in S$ we have $x R x$.

Ex: The relation from the previous example is not reflexive because $(2,2) \notin R$.

Ex: Consider the relation $x R y$ when $5 \mid (x-y)$ on $\mathbb{N}$.
This relation is reflexive. Indeed, for all $x \in \mathbb{N}$, we have $x-x=0=5 \cdot 0$, so $5 \mid x-x$ yields $x R x$.
Hence R is reflexive on $\mathbb{N}$.

Ex: Consider the relation $x R y$ when $5 \mid (x-y+1)$.
This relation is not reflexive. Let $x=0$.

$$x-x+1 = 0-0+1 = 1, \text{ so } 5 \nmid (0-0+1); \text{ thus}$$

$0 \not R 0$. Hence R is NOT reflexive.

↑ "does not relate via R "

NB: In the pictures, "reflexive" means there is a loop at every element: $x \Rightarrow x$.

Obs: $x=y$, then $y=x \dots$

Defn: A relation R on S is symmetric when for all $x, y \in S$ we have xRy implies yRx .

Ex: Let xRy when $5|(x-y)$ on $\mathbb{Z}$. Then

R is symmetric. Let $x, y \in \mathbb{Z}$ and suppose

xRy . Thus $5|(x-y)$, so $x-y=5k$ for some

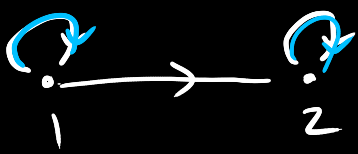
$k \in \mathbb{Z}$. Multiplying by -1 , we have $y-x = -5k = 5(-k)$ ↑

so $5|(y-x)$; thus yRx .

-k $\in \mathbb{Z}$
by closure.

Hence R is symmetric, as claimed. QED

Ex: Consider the relation depicted below:



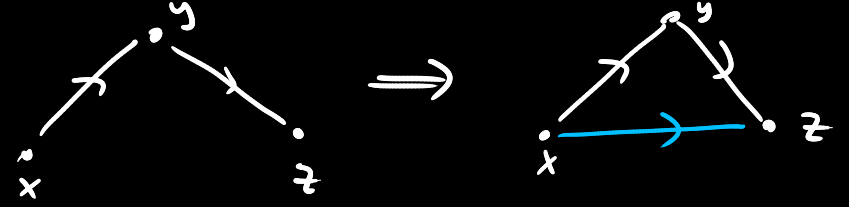
This relation is NOT symmetric:
we have $1R2$, but $2 \nR 1$.

On the other hand, R is reflexive. QED

Point: Reflexive does NOT imply symmetric...

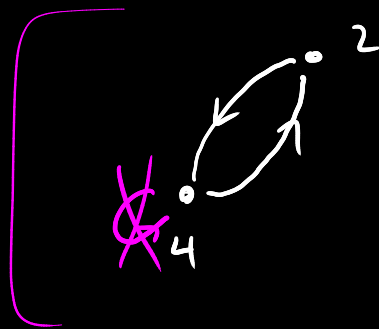
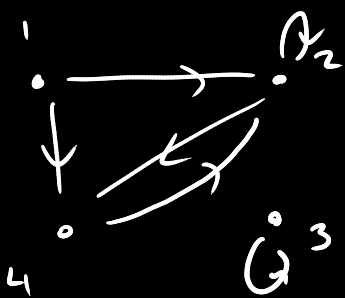
Observation: if $x=y$ and $y=z$, then $x=z$.

Defn: A relation R on set S is transitive when for all $x, y, z \in S$ we have xRy and yRz implies xRz .

Picture: If you see 

Ex: For xRy when $5|(x-y)$ on $\mathbb{Z}$, we note R is transitive. Let $x, y, z \in \mathbb{Z}$ and suppose xRy and yRz . Hence, $5|(x-y)$ and $5|(y-z)$ by definition of R . Thus there are $k, m \in \mathbb{Z}$ with $x-y = 5k$ and $y-z = 5m$. Now we notice $x-z = (x-y) + (y-z) = 5k + 5m = 5(k+m)$. As $k+m \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under addition, we see $5|(x-z)$. Hence, by definition of R , we have xRz . Hence we have shown that for all $x, y, z \in \mathbb{Z}$, if both xRy and yRz , then xRz . Hence R is transitive by definition of transitivity. $\square$

Ex: The relation depicted below is not transitive.



Indeed, $4R2$ and $2R4$, but $4 \not R 4$. ★

Ex: R is reflexive, symmetric and transitive.

$$R = \{(3, 3)\}$$

Reflexive: $\checkmark$

Symmetric: $xRy \Rightarrow x=y=3 \Rightarrow yRx$.

Transitive: $xRy \wedge yRz \Rightarrow x=y=z=3 \Rightarrow xRz$. $\square$

Q: Is $\cdot_x$ reflexive? Symmetric? Transitive?